ON THE DENSITY OF PRIMES WITH A SET OF QUADRATIC RESIDUES OR NON-RESIDUES IN GIVEN ARITHMETIC PROGRESSION

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ABSTRACT. Let $\mathcal{A}$ denote a finite set of arithmetic progressions of positive integers and let $s \geq 2$ be an integer. If the cardinality of $\mathcal{A}$ is at least 2 and U is the union formed by taking certain arithmetic progressions of length s from each element of $\mathcal{A}$, we calculate the natural density of the set of all prime numbers p such that U is a set of quadratic residues (respectively, quadratic non-residues) of p.

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1. Introduction

If p is an odd prime, an integer z is said to be a *quadratic residue* (respectively, *quadratic non-residue*) of p if the equation $x^2 \equiv z \bmod p$ has (respectively, does not have) a solution x in integers. It is a theorem going all the way back to Euler that exactly half of the integers from 1 through $p-1$ are quadratic residues of p, and it is a fascinating problem to investigate the various ways in which these residues are distributed among 1, 2,. . . , $p-1$. In this paper, our particular interest lies in measuring the size of the set of odd primes p such that p has a set of quadratic residues or non-residues that form a union of two or more given arithmetic progressions all of a fixed length.

We begin with a litany of notation and terminology that will be used systematically throughout the rest of this paper. If $m \leq n$ are integers, then $[m, n]$ will denote the set of all integers that are at least m and no greater than n, listed in increasing order, and $[m, +\infty)$ will denote the set of all integers that exceed $m-1$, also listed in increasing order. If $\{a(p)\}$ and $\{b(p)\}$ are sequences of real numbers defined for all primes p in an infinite set S, then we will say that $a(p)$ is (sharply) *asymptotic to* $b(p)$ *as* $p \to +\infty$ *inside* S, denoted as $a(p) \sim b(p)$, if

$$\lim_{\substack{p\to+\infty \\ p\in S}} \frac{a(p)}{b(p)} = 1,$$

and if $S = [1, +\infty)$, we simply delete the phrase "inside S". If A is a set then $|A|$ will denote the cardinality of A, 2^A will denote the set of all subsets of A, $\mathcal{E}(A)$ will denote the set of all nonempty finite subsets of A of even cardinality, and $\emptyset$ will denote the empty set. Finally, we note once and for all that p will always denote a generic odd prime.

In order to state precisely what we wish to do here, it will be convenient to recall one of the principal results from [2]. Let $(m, s) \in [2, +\infty) \times [1, +\infty)$, let $\mathbf{a} = (a_1, \ldots, a_m)$, (respectively, $\mathbf{b} = (b_1, \ldots, b_m)$) be an m-tuple of nonnegative (respectively, positive) integers such that

$(a_i, b_i) \neq (a_j, b_j)$ for $i \neq j$, and let $(\mathbf{a}, \mathbf{b})$ denote the $2m$-tuple $(a_1, \ldots, a_m, b_1, \ldots, b_m)$. We then let $AP(\mathbf{a}, \mathbf{b}; s)$ denote the set

$$\Big\{ \bigcup_{j=1}^{m} \{a_j + b_j(n+i) : i \in [0, s-1]\} : n \in [1, +\infty) \Big\}.$$

If p is an odd prime and $\mathbb{Z}_p$ is the field of p elements, then the Legendre symbol of p defines a real (primitive) multiplicative character $\chi_p : \mathbb{Z}_p \to [-1, 1]$ on $\mathbb{Z}_p$. We take $\varepsilon \in \{-1, 1\}$, let

$$q_\varepsilon(p) = |\{A \in AP(\mathbf{a}, \mathbf{b}; s) \cap 2^{[1,p-1]} : \chi_p(a) = \varepsilon, \text{ for all } a \in A\}|,$$

and note that the value of $q_\varepsilon(p)$ for $\varepsilon = 1$ (respectively, $\varepsilon = -1$) counts the number of elements of $AP(\mathbf{a}, \mathbf{b}; s)$ that are sets of quadratic residues (respectively, non-residues) of p that are located inside $[1, p-1]$.

In [2], the sharp asymptotic behavior of $q_\varepsilon(p)$ as $p \to +\infty$ was determined. It transpires that $q_\varepsilon(p)$ either has an asymptotic limit as $p \to +\infty$ or $q_\varepsilon(p)$ asymptotically oscillates infinitely often between 0 and an asymptotic limit as $p \to +\infty$ through a certain infinite set of primes. In order to more precisely describe this behavior, several ingredients from an appropriate recipe must first be listed. Begin by considering the set B of *distinct* values of the coordinates of $\mathbf{b}$. If we declare the coordinate a_i of $\mathbf{a}$ and the coordinate b_i of $\mathbf{b}$ to *correspond* to each other, then for each $b \in B$, we let $A(b)$ denote the set of all coordinates of $\mathbf{a}$ whose corresponding coordinate of $\mathbf{b}$ is b. We then relabel the elements of B as $b_1, \ldots, b_k$, say, and for each $i \in [1, k]$, set

$$S_i = \bigcup_{a \in A(b_i)} \{ab_i^{-1} + j : j \in [0, s-1]\}.$$

The next ingredient is a certain collection of subsets of $[1, k]$ which is constructed from the sets $S_1, \ldots, S_k$ in the following manner: let

$$\mathcal{K} = \Big\{ \emptyset \neq K \subseteq [1, k] : \bigcap_{i \in K} S_i \neq \emptyset \Big\},$$

$$T(K) = \Big(\bigcap_{i \in K} S_i \Big) \cap \Big(\bigcap_{i \in [1,k] \setminus K} (\mathbf{Q} \setminus S_i) \Big), K \in \mathcal{K},$$

where $\mathbf{Q}$ denotes the set of all rational numbers, and let

$$\mathcal{K}_{\max} = \{K \in \mathcal{K} : T(K) \neq \emptyset\}.$$

The set of subsets of $[1, k]$ that we need is then defined to be the set

$$\Lambda(\mathcal{K}) = \bigcup_{K \in \mathcal{K}_{\max}} \mathcal{E}(K).$$

N.B. $\Lambda(\mathcal{K})$ is empty if and only if the sets $S_1, \ldots, S_k$ are pairwise disjoint.

Suppose that $\Lambda(\mathcal{K})$ is not empty. It will be convenient to declare that p is an *allowable prime* if no element of B has p as a factor. If p is an allowable prime then we define the $(\mathbf{a}, \mathbf{b})$-*signature of* p to be the multiset of ± 1's given by

$$\Big\{ \chi_p\Big(\prod_{i \in I} b_i \Big) : I \in \Lambda(\mathcal{K})) \Big\}$$

and then set $\Pi_+(\mathbf{a},\mathbf{b})$ (respectively, $\Pi_-(\mathbf{a},\mathbf{b})$) equal to the set of all allowable primes p such that the $(\mathbf{a},\mathbf{b})$-signature of p contains only 1's (respectively, contains a -1). For the final ingredients of our recipe, we take

$$b = \max\{b_1, \dots, b_k\},$$
$$\kappa = \Big|\bigcup_{i=1}^{k} S_i\Big|.$$

The asymptotic behavior of $q_\varepsilon(p)$ can now be precisely described. According to Theorem 6.1 of [2],

(i) if either $S_1, \dots, S_k$ are pairwise disjoint or for all $I \in \Lambda(\mathcal{K})$, $\prod_{i\in I} b_i$ is a square, then

$$q_\varepsilon(p) \sim (b \cdot 2^\kappa)^{-1} p \text{ as } p \to +\infty, \text{or}$$

(ii) if there exists $I \in \Lambda(\mathcal{K})$ such that $\prod_{i\in I} b_i$ is not a square, then
(a) $\Pi_+(\mathbf{a},\mathbf{b})$ and $\Pi_-(\mathbf{a},\mathbf{b})$ are both infinite,
(b) $q_\varepsilon(p) = 0$ for all p in $\Pi_-(\mathbf{a},\mathbf{b})$, and
(c) as $p \to +\infty$ inside Π_+,

$$q_\varepsilon(p) \sim (b \cdot 2^\kappa)^{-1} p\ .$$

The problem that is of interest to us in this article stems from the situation present in statement (ii). In that case, sets of quadratic residues and non-residues form inside $AP(\mathbf{a},\mathbf{b};s) \cap 2^{[1,p-1]}$ only for all primes that are sufficiently large inside $\Pi_+(\mathbf{a},\mathbf{b})$, and for no other allowable primes. A natural and interesting question which therefore arises asks: how large can the set $\Pi_+(\mathbf{a},\mathbf{b})$ be and how can we measure its size in an accurate way?

A good way to measure the size of an infinite set Π of primes is to calculate it (natural or absolute) density. If we let P denote the set of all primes then the *density* of Π (in P) is defined to be the limit

$$\lim_{x\to+\infty} \frac{\big|\{p \in \Pi : \ p \le x\}\big|}{\big|\{p \in \mathrm{P} : \ p \le x\}\big|},$$

provided that this limit exists. Roughly speaking, the density of Π measures the "proportion" of the set of all primes that are contained in Π. We will answer the question posed at the end of the previous paragraph by calculating the density of $\Pi_+(\mathbf{a},\mathbf{b})$. Because $\Pi_+(\mathbf{a},\mathbf{b})$ and $\Pi_-(\mathbf{a},\mathbf{b})$ are disjoint sets with only finitely many primes outside of their union, it follows that the density of $\Pi_-(\mathbf{a},\mathbf{b})$ is 1 minus the density of $\Pi_+(\mathbf{a},\mathbf{b})$. Hence a determination of the density of $\Pi_+(\mathbf{a},\mathbf{b})$ also yields a precise measure of the size of the set of primes p such that *no* element of $AP(\mathbf{a},\mathbf{b};s) \cap 2^{[1,p-1]}$ is either a set of quadratic residues or a set of quadratic non-residues of p.

In order to avoid becoming ensnared in technicalities that tend to obscure the essential features of what we wish to study, we will focus our attention on an interesting subclass of $2k$-tuples $(\mathbf{a},\mathbf{b})$. If $k \ge 2$, the coordinates of $\mathbf{b}$ are distinct, and $a_ib_j - a_jb_i \ne 0$ for all $i \ne j$, then we will say that $(\mathbf{a},\mathbf{b})$ is *admissible*. When $(\mathbf{a},\mathbf{b})$ is admissible, the sets $S_1, \dots, S_k$ simplify to

(*) $$S_i = \{a_ib_i^{-1} + j : j \in [0, s-1]\}, i \in [1,k].$$

As we will see in Lemma 3.1 in section 3, the sets $\{b_i : i \in I\}, I \in \Lambda(\mathcal{K})$, determine the primes in $\Pi_+(\mathbf{a}, \mathbf{b})$. The simple structure of the sets in (*), together with the fact that $a_i b_j - a_j b_i \neq 0$ for all $i \neq j$, leads to a very useful combinatorial formula for calculation of the set $\Lambda(\mathcal{K})$, and this formula will hence play a major role in our calculation of the density of $\Pi_+(\mathbf{a}, \mathbf{b})$.

In section 2 we will present this formula for $\Lambda(\mathcal{K})$, together with some other density and combinatorial results that will be required in section 3. In the latter section, the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ will be computed for admissible $2k$-tuples $(\mathbf{a}, \mathbf{b})$ for which certain conditions on the square-free parts of the coordinates of $\mathbf{b}$ are satisfied; these results are the content of Theorems 3.5 and 3.11, the primary results of this paper. We proceed by following a simple strategy: we first decompose $\Pi_+(\mathbf{a}, \mathbf{b})$ into a finite, pairwise disjoint union of certain sets, then use the density results of section 2 to calculate the density of these sets, sum everything up, and, finally, use the combinatorial results of section 2 to evaluate this sum.

2. Preliminaries

In this section we set up the mathematical technology that is required to carry out the calculation of the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ to be performed in section 3. We begin with two lemmas that will be used to determine the densities of various sets. The first lemma is due to Filaseta and Richman [1, Theorem 2] and the second can be found in [3, Theorem 3.3].

Lemma 2.1. *If S is a nonempty finite set of primes and $\varepsilon : S \to \{-1, 1\}$ is a choice of signs for the elements of S then $2^{-|S|}$ is the density of the set $\{p : \chi_p(z) = \varepsilon(z), \text{for all } z \in S\}$.*

The statement of the second lemma requires some preparatory notation. Let F denote the Galois field $[1, +\infty)/2[1, +\infty)$ of 2 elements , let A be a finite nonempty subset of $[1, +\infty)$, let $n = |A|$, and let F^n denote the vector space over F of dimension n. We arrange the elements $a_1 < \cdots < a_n$ of A in increasing order and then define the map $v : 2^A \to F^n$ as follows: if $S \subseteq A$ then the i-th coordinate of $v(S)$ is 1 (respectively, 0) if $a_i \in S$ (respectively, $a_i \notin S$). If $z \in [1, +\infty)$, then we denote by $\pi_{\text{odd}}(z)$ the set of prime factors of z of odd multiplicity.

Lemma 2.2. *If S is a nonempty finite subset of $[1, +\infty), T = S \setminus \{m^2 : m \in [1, +\infty)\}, \mathcal{T} = \{\pi_{odd}(z) : z \in T\}, A = \bigcup\{T : T \in \mathcal{T}\}, n = |A|$, and*

$$d = \text{the dimension of the linear span of } v(\mathcal{T}) \text{ in } F^n,$$

then 2^{-d} is the density of the set $\{p : \chi_p(z) = 1, \text{for all } z \in S\}$.

The next lemma records some simple enumerative combinatorics that will prove useful in section 3.

Lemma 2.3. ([3, *Lemma* 3.2]) *If A is a nonempty finite subset of $[1, +\infty)$, $n = |A|$, $\mathcal{S}$ and $\mathcal{T}$ are disjoint subsets of 2^A and d is the dimension of the linear span of $v(\mathcal{S} \cup \mathcal{T})$ in F^n then the cardinality of the set*

$$\{N \subseteq A : |N \cap S| \text{ is odd, for all } S \in \mathcal{S} \text{ and } |N \cap T| \text{ is even, for all } T \in \mathcal{T}\}$$

is either 0 or 2^{n-d}.

The calculation of the density in section 3 will require a criterion for when the set in the conclusion of Lemma 2.3 is nonempty. In order to state it we recall that the *symmetric difference* $A \triangle B$ of sets A and B is defined as $(A \setminus B) \cup (B \setminus A)$. The symmetric difference

operation is commutative and associative, hence if $\{A_1, \ldots, A_m\}$ is a finite set of sets then the repeated symmetric difference

$$A_1 \triangle \cdots \triangle A_m$$

is unambiguously defined. In fact, one can prove that $A_1 \triangle \cdots \triangle A_m$ is the set

$$\Big\{a \in \bigcup_{i=1}^{m} A_i : |\{A_j : a \in A_j| \text{ is odd}\Big\}. \tag{2.1}$$

The next lemma is a simple reformulation of [3, Proposition 3.5].

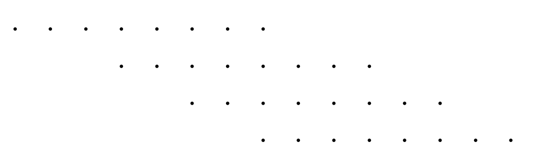
Lemma 2.4. *If A is a nonempty finite set and $\mathcal{S}$ and $\mathcal{T}$ are disjoint subsets of 2^A, with $\emptyset \notin \mathcal{S}$, then the set*

$$\{N \subseteq A : |N \cap S| \textit{ is odd, for all } S \in \mathcal{S} \textit{ and } |N \cap T| \textit{ is even, for all } T \in \mathcal{T}\}$$

is not empty if and only if for each subset U of $\mathcal{S} \cup \mathcal{T} \cup \{\emptyset\}$ of odd cardinality, either the cardinality of $U \cap (\mathcal{T} \cup \{\emptyset\})$ is odd or the repeated symmetric difference of the elements of U is not empty.

We will now present for the set $\Lambda(\mathcal{K})$ that was defined for $2m$-tuples $(\mathbf{a}, \mathbf{b})$ in section 1 a very useful combinatorial formula. The formula requires the idea of an overlap diagram, defined and studied in [2], and so we will discuss that first.

Begin by choosing $(n, s) \in [1, +\infty) \times [2, +\infty)$ and let $\mathbf{g} = (g(1), \ldots, g(n))$ be an n-tuple of positive integers. We use $\mathbf{g}$ to construct the following array of points. In the plane, place s points horizontally one unit apart, and label the j-th point as $(1, j-1)$ for each $j \in [1, s]$. This is *row* 1. Suppose that row i has been defined. One unit vertically down and $g(i)$ units horizontally to the right of the first point in row i, place s points horizontally one unit apart, and label the j-th point as $(i+1, j-1)$ for each $j \in [1, s]$. This is *row* $i+1$. The array of points so formed by these $n+1$ rows is called the *overlap diagram of* $\mathbf{g}$, the sequence $\mathbf{g}$ is called the *gap sequence* of the overlap diagram, and a nonempty set that is formed by the intersection of the diagram with a vertical line is called a *column* of the diagram. N.B. We do not distinguish between the different possible positions in the plane which the overlap diagram may occupy. A typical example with $n = 3, s = 8$, and gap sequence (3, 2, 2) looks like



Next, we need to describe how and where rows overlap in an overlap diagram. Begin by first noticing that if $(g(1), \ldots, g(n))$ is the gap sequence, then row i overlaps row j for $i < j$ if and only if

$$\sum_{r=i}^{j-1} g(r) \leq s - 1;$$

in particular, row i overlaps row $i+1$ if and only if $g(i) \leq s - 1$. Now let $\mathcal{G}$ denote the set of all subsets G of $[1, n]$ such that G is a nonempty set of consecutive integers maximal

with respect to the property that $g(i) \leq s-1$ for all $i \in G$. If $\mathcal{G}$ is empty then $g(i) \geq s$ for all $i \in [1,n]$, and so there is no overlap of rows in the diagram. Otherwise there exists $m \in [1, 1+[(n-1)/2]]$ and strictly increasing sequences $(l_1, \dots, l_m)$ and $(M_1, \dots, M_m)$ of positive integers, uniquely determined by the gap sequence of the diagram, such that $l_i \leq M_i$ for all $i \in [1,m], 1+M_i \leq l_{i+1}$ if $i \in [1, m-1]$, and

$$\mathcal{G} = \{[l_i, M_i] : i \in [1,m]\}.$$

In fact, $l_{i+1} > 1 + M_i$ if $i \in [1, m-1]$, lest the maximality of the elements of $\mathcal{G}$ be violated. It follows that the intervals of integers $[l_i, 1+M_i], i \in [1,m]$, are pairwise disjoint.

The set $\mathcal{G}$ can now be used to locate the overlap between rows in the overlap diagram like so: for $i \in [1,m]$, let

$$B_i = [l_i, 1+M_i],$$

and set

$$\mathcal{B}_i = \text{the set of all points in the overlap diagram whose labels are in } B_i \times [0, s-1].$$

We refer to $\mathcal{B}_i$ as the *i-th block* of the overlap diagram, to the interval of integers B_i as the *support of* $\mathcal{B}_i$, and to the sequence $(g(j) : j \in [l_i, M_i])$ as the *gap sequence* of $\mathcal{B}_i$. Thus the blocks of the diagram are precisely the regions in the diagram in which rows overlap.

Our intent now is to use certain overlap diagrams defined by means of an admissible $2k$-tuple to calculate $\Lambda(\mathcal{K})$. We fix an admissible $2k$-tuple $(\mathbf{a}, \mathbf{b})$ and proceed to construct this series of overlap diagrams.

Begin by letting $q_i = a_i/b_i$ for $i \in [1,k]$; without loss of generality, we suppose that the coordinates of $\mathbf{a}$ and $\mathbf{b}$ are indexed so that $q_i < q_{i+1}$ for each $i \in [1, k-1]$. Consider now the set $Q(\mathbf{a}, \mathbf{b})$ of all elements (i,j) of $[1,k] \times [1,k]$ such that $i \neq j$ and $b_i b_j$ divides $a_i b_j - a_j b_i$, with quotient $q(i,j)$, say. In light of the fact that $a_i b_j - a_j b_i \neq 0$ for $i \neq j$, it follows that if $(i,j) \in Q(\mathbf{a}, \mathbf{b})$ then $q(i,j) \neq 0$. Because $\Lambda(\mathcal{K})$ is empty if and only if $Q(\mathbf{a}, \mathbf{b})$ contains no elements (i,j) such that $|q(i,j)| \leq s-1$, we need only calculate e when $Q(\mathbf{a}, \mathbf{b})$ contains elements of this type. Thus, suppose that this is so.

Let π denote the canonical projection of $[1,k] \times [1,k]$ onto its left factor. If $(i,j) \in \pi(Q(\mathbf{a}, \mathbf{b})) \times \pi(Q(\mathbf{a}, \mathbf{b}))$ and we declare that $i \simeq j$ if either $i = j$ or $(i,j) \in Q(\mathbf{a}, \mathbf{b})$, then $\simeq$ defines an equivalence relation on $\pi(Q(\mathbf{a}, \mathbf{b}))$.

We will now use the equivalence classes of $\simeq$ to construct a series of overlap diagrams. Let E be an equivalence class such that $|q(i,j)| \leq s-1$ for some $(i,j) \in E \times E$. We note that the elements of the set $\{q_i : i \in E\}$ are listed in increasing order with increasing i and $q_i - q_j = q(i,j)$ for $i, j \in E$ with $i \neq j$. Next, consider the nonempty and pairwise disjoint family of all subsets Q of $\{q_i : i \in E\}$ such that $|Q| \geq 2$ and Q is maximal with respect to the property that the distance between consecutive elements of Q does not exceed $s-1$. The distances between consecutive elements of Q are equal to certain positive quotients $q(i,j)$. We index those positive quotients as $(q_Q(i) : i \in [1, |Q|-1])$, and then let $\mathcal{D}(Q)$ denote the overlap diagram of this $(|Q|-1)$-tuple. Because $q_Q(i) \leq s-1$ for all $i \in [1, |Q|-1]$, $\mathcal{D}(Q)$ consists of a single block.

Using a suitable positive integer v, we index all of the sets Q that arise from all of the equivalence classes in the previous construction as $Q_1, \dots, Q_v$ and then define the *quotient diagram* of $(\mathbf{a}, \mathbf{b})$ to be the v-tuple of overlap diagrams $(\mathcal{D}(Q_n) : n \in [1,v])$. The overlap diagrams $\mathcal{D}(Q_1), \dots, \mathcal{D}(Q_v)$ are called the *blocks* of the quotient diagram.

Let $v \in [1, +\infty)$ and for each $n \in [1, v]$, let $\mathcal{D}(n)$ be a fixed but arbitrary overlap diagram with k_n rows, $k_n \geq 2$, and gap sequence $(d(i, n) : i \in [1, k_n - 1])$, with no gap exceeding $s-1$. Let $k_0 = 0, k = \sum_n k_n$. We will now exhibit infinitely many admissible $2k$-tuples $(\mathbf{a}, \mathbf{b})$ whose quotient diagram is $\Delta = (\mathcal{D}(n) : n \in [1, v])$. This is done by taking the $(k-1)$-tuple $(d_1, \ldots, d_{k-1})$ in the following lemma to be

$$d_i = \begin{cases} d\Big(i - \sum_0^n k_j, n+1\Big), \text{ if } n \in [0, v-1] \text{ and } i \in \Big[1 + \sum_0^n k_j, -1 + \sum_0^{n+1} k_j\Big], \\ \qquad\qquad\qquad\qquad s, \text{ elsewhere}, \end{cases}$$

and then letting $(\mathbf{a}, \mathbf{b})$ be any $2k$-tuple obtained from the construction in the lemma.

Lemma 2.5. *For $k \in [2, +\infty)$, let $(d_1, \ldots, d_{k-1})$ be a $(k-1)$-tuple of positive integers. Define k-tuples $(a_1, \ldots, a_k), (b_1, \ldots, b_k)$ of positive integers inductively as follows: let (a_1, b_1) be arbitrary, and if $i > 1$ and (a_i, b_i) has been defined, choose $t_i \in [2, +\infty)$ and set*

$$a_{i+1} = t_i(a_i + d_i b_i), \quad b_{i+1} = t_i b_i.$$

Then

$$a_i b_j - a_j b_i = \Big(\sum_{r=j}^{i-1} d_r\Big) b_i b_j, \textit{ for all } i > j.$$

If one chooses b_1 and all subsequent $t_i's$ in the construction of Lemma 2.5 to be distinct primes, then one obtains infinitely many admissible $2k$-tuples $(\mathbf{a}, \mathbf{b})$ with given quotient diagram Δ such that $\Pi_+(\mathbf{a}, \mathbf{b})$ and $\Pi_-(\mathbf{a}, \mathbf{b})$ are both infinite. As we shall see in section 3, this shows that there are infinitely many admissible $2k$-tuples with a fixed but arbitrary quotient diagram such that the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ and $\Pi_-(\mathbf{a}, \mathbf{b})$ are both positive.

The quotient diagram $\mathcal{D}$ of $(\mathbf{a}, \mathbf{b})$ will now be used to give the promised formula for $\Lambda(\mathcal{K})$. In order to do that, a certain labeling of the points of $\mathcal{D}$ is required, which we describe first. N.B. This labeling will in general be different from the labeling of points of an overlap diagram that was used to define the blocks of the overlap diagram. Let $Q_1, \ldots, Q_v$ be the subsets of $\{q_1, \ldots, q_k\}$ that determine the sequence of overlap diagrams $\mathcal{D}(Q_1), \ldots, \mathcal{D}(Q_v)$ which constitute $\mathcal{D}$, and then find the subset J_n of $[1, k]$ such that $Q_n = \{q_j : j \in J_n\}$, with $j \in J_n$ listed in increasing order.The overlap diagram $\mathcal{D}(Q_n)$ consists of $|J_n|$ rows, with each row containing s points. If $i \in [1, |J_n|]$ is taken in increasing order then there is a unique element j of J_n such that the i-th element of Q_n is q_j. Proceeding from left to right in each row, we now take $l \in [1, s]$ and label the l-th point of row i in $\mathcal{D}(Q_n)$ as $(j, l-1)$.

Next let C denote a column of one of the diagrams $\mathcal{D}(Q_n)$ which constitute $\mathcal{D}$. We identify C with the subset of $[1, k] \times [0, s-1]$ defined by

$$\{(i, j) \in [1, k] \times [0, s-1] : (i, j) \text{ is the label of a point in } C\}, \tag{2.2}$$

let $\mathcal{C}_n$ denote the set of all subsets of $[1, k] \times [0, s-1]$ which arise from all such identifications, and then set $\mathcal{C} = \bigcup_n \mathcal{C}_n$. If θ denotes the projection of $[1, k] \times [0, s-1]$ onto $[1, k]$ then it can be shown that $K \in \mathcal{K}_{\max}$ if and only if there exists a $T \in \mathcal{C}$ such that $K = \theta(T)$. Hence the following lemma is in hand:

Lemma 2.6. *If $(\boldsymbol{a}, \boldsymbol{b})$ is an admissible $2k$-tuple, $\mathcal{C}$ is the set of subsets of $[1, k] \times [0, s-1]$ defined above using the sets in (2.2), and θ is the canonical projection of $[1, k] \times [0, s-1]$*

onto $[1,k]$ *then*

$$\Lambda(\mathcal{K}) = \bigcup_{X\in\mathcal{C}} \mathcal{E}(\theta(X)).$$

3. The density of $\Pi_+(\mathbf{a},\mathbf{b})$

We begin with the following general situation and then specialize it to the case of interest to us here, namely that of the set $\Pi_+(\mathbf{a},\mathbf{b})$. The ability to explain our reasoning concisely in the sequel will be enhanced if we employ the following notation: if A is a nonempty finite set and $\mathcal{A} \subseteq 2^A$, then $\mathcal{U}(\mathcal{A})$ will denote the set formed by the union of all the elements of $\mathcal{A}$, and $\mathcal{P}(A,2)$ will denote the set of all 2-block partitions of A.

Let $\mathbf{S}$ denote a set $\{S_1,\dots,S_m\}$ of nonempty, finite subsets of $[1,+\infty)$ such that for each i, all elements of S_i are square-free.. We will say that p is an *allowable prime* (with respect to $\mathbf{S}$) if no element of $\bigcup_i S_i$ has p as a factor, and then we will let

$$\Pi_+(\mathbf{S})$$

denote the set of all allowable primes such that for all i, S_i is either a set of quadratic residues of p or a set of quadratic non-residues of p. This is a generalization of the set $\Pi_+(\mathbf{a},\mathbf{b})$, as is clear from the following lemma:

Lemma 3.1. ([2, *Lemma* 4.1]) *If* $(\boldsymbol{a},\boldsymbol{b})$ *is a* $2m$*-tuple as defined in the begining of section 1 and* $\{b_1,\dots,b_k\}$ *is the set of distinct values of the coordinates of* $\boldsymbol{b}$ *then* $\Pi_+(\boldsymbol{a},\boldsymbol{b})$ *consists precisely of all primes allowable with respect to* $\{b_1,\dots,b_k\}$ *such that each of the sets* $\{b_i : i\in I\}, I\in\Lambda(\mathcal{K})$, *is either a set of quadratic residues of* p *or a set of quadratic non-residues of* p.

For each $i\in[1,m]$, let X_i^+ (respectively, X_i^-) denote the set of all allowable primes p such that χ_p, when restricted to S_i, is identically 1, (respectively, is identically -1). Let

$$M_0 = \{i\in[1,m] : 1\notin S_i\},$$
$$M_1 = \{i\in[1,m] : 1\in S_i\}.$$

Because X_i^- is empty if $i\in M_1$, it follows that

$$\begin{aligned}\Pi_+(\mathbf{S}) &= \bigcap_{i=1}^{m} \Big(X_i^+\cup X_i^-\Big)\\ &= \Big(\bigcap_{i\in M_1} X_i^+\Big)\cap\Big(\bigcap_{i\in M_0}\Big(X_i^+\cap X_i^-\Big)\Big).\end{aligned}$$

Let $M_0 = \{i_j : j\in[1,\mu]\}$, where $\mu = |M_0|$. Because X_i^+ is disjoint from X_i^-, for all i, we may write

$$\bigcap_{i\in M_0}\Big(X_i^+\cap X_i^-\Big) = \bigcup_{(Z_1,\dots,Z_\mu):Z_j\in\{X_{i_j}^+,\ X_{i_j}^-\},\forall j} Z_1\cap\cdots\cap Z_\mu,$$

and this union is pairwise disjoint. We rearrange this union so that $\Pi_+(\mathbf{S})$ is the pairwise disjoint union of the sets

$$\bigcap_{1=1}^{m} X_i^+, \tag{3.1}$$

$$\Big(\bigcap_{i\in M_1} X_i^+\Big)\cap\Big(\bigcap_{i\in M_0} X_i^-\Big), \tag{3.2}$$

$$\Big(\bigcap_{i\in M_1\cup P_1} X_i^+\Big)\cap\Big(\bigcap_{i\in P_2} X_i^-\Big), \tag{3.3}$$

$$\Big(\bigcap_{i\in P_1} X_i^-\Big)\cap\Big(\bigcap_{i\in M_1\cup P_2} X_i^+\Big),\ \{P_1,P_2\}\in\mathcal{P}(M_0,2), \tag{3.4}$$

In order to calculate the density of $\Pi_+(\mathbf{S})$, it hence suffices to calculate the densities of each of these sets and then add everything up.

We proceed to do precisely that. Observe first that $\bigcap_{i=1}^m X_i^+$ is the set of all allowable primes p such that $\bigcup_{i=1}^m S_i$ is a set of quadratic residues of p. Lemma 2.2 hence provides a way to calculate the density of this set. After letting $\pi(z)$ denote the set of prime factors of a square-free integer z, we set

$$\Pi=\bigcup\ \Big\{\pi(z) : z\in\bigcup_i S_i\Big\},$$

$$n=|\Pi|,\ \text{and},$$

$$\mathcal{S}_i=\{\pi(z) : z\in S_i\},\ i\in[1,m].$$

If F is the Galois field of 2 elements and $v : 2^\Pi\to F^n$ is the bijection defined in section 2, then we conclude from Lemma 2.2 that if

$$d=\text{the dimension of the linear span of }\ v\Big(\Big(\bigcup_i \mathcal{S}_i\Big)\setminus\{\emptyset\}\Big)\ \text{in}\ F^n$$

then

$$\text{the density of} \bigcap_{i=1}^m X_i^+ \text{ is } 2^{-d}. \tag{3.5}$$

The next step in our calculation is to compute the density of each set in (3.2)-(3.4). Let $\{P_1,P_2\}$ be a partition of M_0 . We first decompose each of these sets into a useful pairwise disjoint union. Toward that end, let

$$\begin{aligned}\mathcal{N}(M_0,M_1)=\{N\subseteq\Pi : |N\cap S| &\text{ is even, for all } S\in \textstyle\bigcup_{i\in M_1}\ \mathcal{S}_i\\ &\text{and } |N\cap S| \text{ is odd, for all } S\in\textstyle\bigcup_{i\in M_0}\ \mathcal{S}_i\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_e(\{P_1,P_2\})=\{N\subseteq\Pi : |N\cap S| &\text{ is even, for all } S\in \textstyle\bigcup_{i\in M_1\cup P_1} \mathcal{S}_i\\ &\text{and} |N\cap S| \text{ is odd, for all } S\in\textstyle\bigcup_{i\in P_2} \mathcal{S}_i\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_o(\{P_1,P_2\})=\{N\subseteq\Pi : |N\cap S| &\text{ is odd, for all } S\in \textstyle\bigcup_{i\in P_1} \mathcal{S}_i\\ &\text{and} |N\cap S| \text{ is even, for all } S\in\textstyle\bigcup_{i\in M_1\cup P_2} \mathcal{S}_i\}.\end{aligned}$$

If for a prime p we set

$$N(p) = \{q \in \Pi : \chi_p(q) = -1\},$$

then

$$\Big(\bigcap_{i\in M_1} X_i^+\Big) \cap \Big(\bigcap_{i\in M_0} X_i^-\Big) = \bigcup_{N\in\mathcal{N}(M_0,M_1)} \{p : N(p) = N\},$$

$$\Big(\bigcap_{i\in M_1\cup P_1} X_i^+\Big) \cap \Big(\bigcap_{i\in P_2} X_i^-\Big) = \bigcup_{N\in\mathcal{N}_e(\{P_1,P_2\})} \{p : N(p) = N\},$$

$$\Big(\bigcap_{i\in P_1} X_i^-\Big) \cap \Big(\bigcap_{i\in M_1\cup P_2} X_i^+\Big) = \bigcup_{N\in\mathcal{N}_o(\{P_1,P_2\})} \{p : N(p) = N\},$$

and each of these unions is pairwise disjoint. Now by virtue of Lemma 2.1, the density of each set in these three unions is 2^{-n}. Observe next that the set in (3.2) is nonempty only if

$$\Big(\bigcup_{i\in M_0} S_i\Big) \cap \Big(\bigcup_{i\in M_1} S_i\Big) = \emptyset,$$

and because the elements of the sets S_i are square-free it follows that this holds if and only if

$$\Big(\bigcup_{i\in M_0} \mathcal{S}_i\Big) \cap \Big(\bigcup_{i\in M_1} \mathcal{S}_i\Big) = \emptyset.$$

Consequently, after observing that the dimension of the linear span of

$$v\Big(\Big(\Big(\bigcup_{i\in M_0} \mathcal{S}_i\Big) \cup \Big(\bigcup_{i\in M_1} \mathcal{S}_i\Big)\Big) \setminus \{\emptyset\}\Big) = v\Big(\Big(\bigcup_{i} \mathcal{S}_i\Big) \setminus \{\emptyset\}\Big)$$

in F^n is d, we conclude from Lemma 2.3 that the cardinality of $\mathcal{N}(M_0, M_1)$ is either 2^{n-d} or 0. Following a similar line of reasoning, we find that the cardinality of $\mathcal{N}_e(\{P_1, P_2\})$ and, respectively, $\mathcal{N}_o(\{P_1, P_2\}$, is also either 2^{n-d} or 0. It follows that

$$\text{(3.6)} \qquad \text{the density of } \Big(\bigcap_{i\in M_1} X_i^+\Big) \cap \Big(\bigcap_{i\in M_0} X_i^-\Big) = \begin{cases} 2^{-d}, & \text{if } \mathcal{N}(M_0, M_1) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{(3.7)} \qquad \text{the density of } \Big(\bigcap_{i\in M_1\cup P_1} X_i^+\Big) \cap \Big(\bigcap_{i\in P_2} X_i^-\Big) = \begin{cases} 2^{-d}, & \text{if } \mathcal{N}_e(\{P_1, P_2\}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{(3.8)} \qquad \text{the density of } \Big(\bigcap_{i\in P_1} X_i^-\Big) \cap \Big(\bigcap_{i\in M_1\cup P_2} X_i^+\Big) = \begin{cases} 2^{-d}, & \text{if } \mathcal{N}_o(\{P_1, P_2\}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Summing the densities from (3.5)-(3.8) now yields the following lemma:

Lemma 3.2. *The density of* $\Pi_+(\boldsymbol{S})$ *is*

$$2^{-d}\Big(1 + \varepsilon + \big|\{\{P_1, P_2\} \in \mathcal{P}(M_0, 2) : \mathcal{N}_e(\{P_1, P_2\}) \neq \emptyset\}\big| + \\ \big|\{\{P_1, P_2\} \in \mathcal{P}(M_0, 2) : \mathcal{N}_o(\{P_1, P_2\}) \neq \emptyset\}\big|\Big),$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } \mathcal{N}(M_0, M_1) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

At this juncture, the combinatorial parameters that occur in Lemma 3.2 are rather daunting to compute, so as to proceed further, we specialize to the case which is obtained when Lemma 3.2, by way of Lemma 3.1, is applied to an admissible $2k$-tuple $(\mathbf{a}, \mathbf{b})$. The first thing to be done is to produce a set $\mathbf{S}$ that can be used in Lemma 3.2 to find the density of $\Pi_+(\mathbf{a}, \mathbf{b})$. We will proceed as follows: assume to begin with that at least one square-free part of the coordinates of $\mathbf{b}$ is not 1; otherwise all of these coordinates are squares, whence the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ is clearly 1. Let σ_i denote the square-free part of $b_i, i \in [1, k]$. For each element I of $\Lambda(\mathcal{K})$, we let $S(I)$ denote the set formed from the integers σ_i for $i \in I$ and then we choose a nonempty subset $Z(I)$ of I such that

$$S(I) = \{\sigma_i : i \in Z(I)\}.$$

If $\mathbf{I}$ denotes the set of subscripts i that index the distinct square-free parts on the list $\sigma_1, \ldots, \sigma_k$ then $Z(I)$ is not contained in $\mathbf{I}$ only if $Z(I) = \{i_0\}$ and $i_0 \notin \mathbf{I}$. Because $\Pi_+(\mathbf{a}, \mathbf{b})$ is unaffected by the elements I of $\Lambda(\mathcal{K})$ for which $S(I)$ is a singleton, if we hence remove all such elements from $\Lambda(\mathcal{K})$, then for each set I in the set $\mathcal{I}$ of elements of $\Lambda(\mathcal{K})$ which remain, it follows that $|S(I)| \geq 2$ and $Z(I) \subseteq \mathbf{I}$.

Next, on $\mathcal{I}$, we define an equivalence relation $\approx$ as follows: if $(I, J) \in \mathcal{I} \times \mathcal{I}$, then declare that $I \approx J$ if $S(I) = S(J)$. Select one representative element of $\mathcal{I}$ from each equivalence class of $\approx$ and let $\Lambda'(\mathcal{K})$ denote the set of elements so chosen. The sets $S(I)$ for $I \in \Lambda'(\mathcal{K})$ all have cardinality at least 2 and are distinct, and it follows from Lemma 3.1 that $\Pi_+(\mathbf{a}, \mathbf{b})$ consists precisely of all primes p allowable with respect to $\{b_1, \ldots, b_k\}$ such that each of the sets $S(I)$ for $I \in \Lambda'(\mathcal{K})$ is either a set of quadratic residues of p or a set of quadratic non-residues of p. Our intension is to use Lemma 3.2 to calculate the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ by letting the set $\{S(I) : I \in \Lambda'(\mathcal{K})\}$ play the role of the set $\mathbf{S}$.

We now impose the following arithmetic condition on the square-free parts of the coordinates of $\mathbf{b}$. Let

$$\Sigma = \bigcup_{I \in \Lambda'(\mathcal{K})} Z(I).$$

We assume that the sets $\pi(\sigma_i), i \in \Sigma$, can be totally ordered by inclusion: this means that there exists a bijection ν of $[1, |\Sigma|]$ onto Σ such that

$$\sigma_{\nu(i+1)} \text{ is divisible by } \sigma_{\nu(i)}, \ i \in [1, |\Sigma|]. \tag{3.9}$$

All of the $2k$-tuples that are produced from the construction in Lemma 2.5 satisfy this condition, hence the admissible $2k$-tuples which satisfy (3.9) exhibit all of the possible block structures that can arise in a quotient diagram. As we shall see, the number of blocks in a quotient diagram is an important ingredient in the calculation of the density of $\Pi_+(\mathbf{a}, \mathbf{b})$. We note incidentally that the indexing used in condition (3.9) may differ from the indexing of the σ_i's that is used to define $\Lambda'(\mathcal{K})$, but this will cause no difficulties.

We proceed to apply Lemma 3.2 with $\mathbf{S} = \{S(I) : I \in \Lambda'(\mathcal{K})\}$. In order to do that, we study first the sets which determine the values of the combinatorial parameters in Lemma 3.2. For each $I \in \Lambda'(\mathcal{K})$, set

$$\mathcal{S}(I) = \{\pi(\sigma) : \sigma \in S(I)\}.$$

Now let

$$\mathcal{M}_0 = \{I \in \Lambda'(\mathcal{K}) : 1 \notin S(I)\},$$

$$\mathcal{M}_1 = \{I \in \Lambda'(\mathcal{K}) : 1 \in S(I)\}.$$

N.B. Upon replacing the sets $[1, m]$, $\mathbf{S}$, M_0 and M_1 defined above by, respectively, $\Lambda'(\mathcal{K})$, $\{S(I) : I \in \Lambda'(\mathcal{K})\}$, $\mathcal{M}_0$, and $\mathcal{M}_1$, we define $\mathcal{N}(\mathcal{M}_0, \mathcal{M}_1)$, $\mathcal{N}_e(\{P_1, P_2\})$ and $\mathcal{N}_o(\{P_1, P_2\})$ accordingly. We set

$$\mathcal{T}(\mathcal{M}_i) = \bigcup_{I \in \mathcal{M}_i} \mathcal{S}(I),\ i \in [0, 1],$$

and suppose that

$$\mathcal{T}(\mathcal{M}_0) \cap \mathcal{T}(\mathcal{M}_1) = \emptyset.$$

We will use Lemma 2.4 with

$$A = \bigcup_{i \in \Sigma} \pi(\sigma_i),$$

$$\mathcal{S} = \mathcal{T}(\mathcal{M}_0),\ \text{and}$$

$$\mathcal{T} = \mathcal{T}(\mathcal{M}_1),$$

to prove that

$$\mathcal{N}(\mathcal{M}_0, \mathcal{M}_1) \neq \emptyset.$$

Hence, let U be a subset of

$$\mathcal{T}(\mathcal{M}_0) \cup \mathcal{T}(\mathcal{M}_1) \cup \{\emptyset\} = \big\{\pi(\sigma_i) : i \in \Sigma\big\} \cup \{\emptyset\}$$

of odd cardinality such that the cardinality of

$$U \cap \Big(\mathcal{T}(\mathcal{M}_1) \cup \{\emptyset\}\Big)$$

is even. Then $U \neq \emptyset \neq U \setminus \{\emptyset\}$. We must prove that the repeated symmetric difference of the sets $X \in U$ is nonempty, and so, in light of equation (2.1) of section 1, we must prove that

$$\Xi = \{q \in A : \big|X \in U \setminus \{\emptyset\} : q \in X\big| \text{ is odd}\} \neq \emptyset.$$

In order to do that, note first that by virtue of (3.9), the sets $\pi(\sigma_i), i \in \Sigma$, are distinct and totally ordered by inclusion. Hence if we let $\alpha = \{i : \pi(\sigma_i) \in U\}$ and let $\pi(\sigma_M)$ be the element of U that is maximal relative to inclusion then

$$\pi(\sigma_M) \setminus \Big(\bigcup_{i \in \alpha \setminus \{M\}} \pi(\sigma_i)\Big)$$

is not empty and is a subset of Ξ.

A similar argument also shows that if $\{P_1, P_2\} \in \mathcal{P}(\mathcal{M}_0, 2)$ and

$$\Big(\bigcup_{I \in \mathcal{M}_1 \cup P_1} \mathcal{S}(I)\Big) \cap \Big(\bigcup_{I \in P_2} \mathcal{S}(I)\Big) = \emptyset$$

then

$$\mathcal{N}_e(\{P_1, P_2\}) \neq \emptyset$$

and if

$$\Big(\bigcup_{I\in P_1}\mathcal{S}(I)\Big)\cap\Big(\bigcup_{I\in\mathcal{M}_1\cup P_2}\mathcal{S}(I)\Big)=\emptyset$$

then

$$\mathcal{N}_o(\{P_1,P_2\})\neq\emptyset.$$

Due to the fact that all σ_i are square-free, we conclude that $\mathcal{N}(\mathcal{M}_0,\mathcal{M}_1)\neq\emptyset$ if and only if

$$\Big(\bigcup_{I\in\mathcal{M}_0}Z(I)\Big)\cap\Big(\bigcup_{I\in\mathcal{M}_1}Z(I)\Big)=\emptyset,$$

$\mathcal{N}_e(\{P_1,P_2\})\neq\emptyset$ if and only if

$$\Big(\bigcup_{I\in\mathcal{M}_1\cup P_1}Z(I)\Big)\cap\Big(\bigcup_{I\in P_2}Z(I)\Big)=\emptyset, \tag{3.10}$$

and $\mathcal{N}_o(\{P_1,P_2\})\neq\emptyset$ if and only if

$$\Big(\bigcup_{I\in P_1}Z(I)\Big)\cap\Big(\bigcup_{I\in\mathcal{M}_1\cup P_2}Z(I)\Big)=\emptyset. \tag{3.11}$$

If we hence let $n=|A|$, $v:2^A\to F^n$ be the bijection that is defined in section 2, let d denote the dimension of the linear span in F^n of the set $\{v(\pi(\sigma_i)):i\in\Sigma\}\setminus\{0\}$, and let

$$\mathcal{P}_\emptyset(\mathcal{M}_0,2)=\big\{\{P_1,P_2\}\in\mathcal{P}(\mathcal{M}_0,2):\text{(3.10) and (3.11) hold for }\{P_1,P_2\}\big\},$$

then Lemma 3.2 implies that

$$\text{the density of }\Pi_+(\mathbf{a},\mathbf{b})=2^{-d}\big(1+\varepsilon+2\big|\mathcal{P}_\emptyset(\mathcal{M}_0,2)\big|\big), \tag{3.12}$$

where

$$\varepsilon=\begin{cases}1, \text{ if } \Big(\bigcup_{I\in\mathcal{M}_0}Z(I)\Big)\cap\Big(\bigcup_{I\in\mathcal{M}_1}Z(I)\Big)=\emptyset,\\ \qquad 0, \text{ otherwise.}\end{cases} \tag{3.13}$$

We turn now to the calculation of the parameters in (3.12), and in order to do that in as perspicuous a manner as possible, it is instructive to treat first the case in which the square-free parts of the coordinates of $\mathbf{b}$ are distinct. We have then that $\Lambda'(\mathcal{K})=\Lambda(\mathcal{K})$ and $S(I)=I$ for all $I\in\Lambda(\mathcal{K})$. Suppose that the quotient diagram of $(\mathbf{a},\mathbf{b})$ has m blocks $(\mathcal{D}_1\ldots,\mathcal{D}_m)$. Let

$$\Lambda_n(\mathcal{K})=\bigcup_{X\in\mathcal{C}_n}\mathcal{E}(\theta(X)),\ n\in[1,m],$$

where $\mathcal{C}_n$ is the set of columns of $\mathcal{D}_n$ as defined according to (2.2) in section 2. It follows from the construction of the quotient diagram and Lemma 2.6 that if we let

$$D_n=\mathcal{U}(\Lambda_n(\mathcal{K})),\ n\in[1,m],$$

then

$$D_i\cap D_j=\emptyset \text{ for } i\neq j \tag{3.14}$$

and

$$\Lambda(\mathcal{K})=\bigcup_n\Lambda_n(\mathcal{K}). \tag{3.15}$$

We also note that
$$\Sigma = \bigcup_i D_i.$$

Suppose first that $\mathcal{M}_1$ is empty, i.e., no square-free part of b_i for $i \in \Sigma$ is 1. As a consequence of (3.13), it follows that

$$\varepsilon = 1. \tag{3.16}$$

Turning next to the parameter d, with $n = |A|$, we first recall that the *support* $\mathrm{supp}(v)$ of a vector $v = (v(1), \ldots, v(n)) \in F^n$ is the set $\{i \in [1, n] : v(i) = 1\}$ and then observe that a nonempty subset V of $F^n \setminus \{0\}$ is linearly independent over F if and only if for each nonempty subset S of V, the repeated symmetric difference of the sets $\mathrm{supp}(v), v \in S$, is not empty. We have that
$$\mathrm{supp}(v(\pi(\sigma_i))) = \pi(\sigma_i),\ i \in \Sigma,$$
and the set $\{\pi(\sigma_i) : i \in \Sigma\}$ is totally ordered by inclusion. It follows that $\{v(\pi(\sigma_i)) : i \in \Sigma\}$ is linearly independent over F, and because this set does not contain 0, we conclude that

$$d = |\Sigma|. \tag{3.17}$$

It remains to determine the cardinality of the set $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$ in the third term of the sum on the right-hand side of (3.12). Toward that end, we begin by observing that in this case,

$$\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2) = \big\{\{P_1, P_2\} \in \mathcal{P}(\Lambda(\mathcal{K}), 2) : \mathcal{U}(P_1) \cap \mathcal{U}(P_2) = \emptyset\big\}. \tag{3.18}$$

That

$$\big|\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)\big| = 2^{m-1} - 1 \tag{3.19}$$

is now an immediate consequence of the next lemma.

Lemma 3.3. *The set $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$ consists precisely of all sets of the form*
$$\Big\{ \bigcup_{i \in Q_1} \Lambda_i(\mathcal{K}),\ \bigcup_{i \in Q_2} \Lambda_i(\mathcal{K}) \Big\}$$
where $\{Q_1, Q_2\}$ varies throughout the set $\mathcal{P}([1, m], 2)$.

Proof. That every set in the conclusion of Lemma 3.3 is in $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$ is an immediate consequence of (3.14), (3.15), and (3.18).

Suppose first that the quotient diagram of $(\mathbf{a}, \mathbf{b})$ consists of a single block $\mathcal{D}$, say. We proceed to construct an element $\{P_1, P_2\}$ of $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$. The following terminology will be an auxilliary to our analysis: if C is a column of one of the blocks $\mathcal{D}_n$ in the quotient diagram of an admissible $2k$-tuple then we will say that C *hangs from* row i *in* $\mathcal{D}_n$ if the top point of C is in row i. We may assume with no loss of generality that for some integer n, the rows of $\mathcal{D}$ are labeled from 1 to n. The columns hanging from row 1 have cardinality either 1 or 2 and there is at least one column of cardinality 2. Hence we place $[1, 2]$ in P_1. It follows that if $n = 2$ then $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$ is empty. Assume that $n > 2$. Observe now that for each of the rows from 2 through $n - 1$, there is at least one column hanging from that row which has cardinality at least 2; otherwise there is a coordinate of the gap sequence of $\mathcal{D}$ that exceeds $s - 1$, which is impossible by construction of the quotient diagram. Thus for all $i \in [2, n-1]$,
$$\max\{|C| : C \text{ is a column hanging from row } i\} \geq\ 2,$$

and so

$$\big\{\{i, i+1\} : i \in [2, n-1]\big\} \subseteq \Lambda(\mathcal{K}). \tag{3.20}$$

The set $[1, 2]$ is already contained in $\mathcal{U}(P_1)$, hence suppose inductively that $[1, i] \subseteq \mathcal{U}(P_1)$. Because of (3.20) and the fact that $\mathcal{U}(P_1)$ and $\mathcal{U}(P_2)$ must be disjoint, we must place $\{i, i+1\}$ in P_1, and so $[1, i+1] \subseteq \mathcal{U}(P_1)$. Hence $\mathcal{U}(P_1) = [1, n]$; we conclude that $\mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$ is empty.

Suppose next that in general the quotient diagram of $(\mathbf{a}, \mathbf{b})$ consists of an m-tuple of blocks $(\mathcal{D}_1, \dots, \mathcal{D}_m)$ with $m \geq 2$. Let $\{P_1, P_2\} \in \mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2)$. If $P_1 \cap \Lambda_n(\mathcal{K})$ and $P_2 \cap \Lambda_n(\mathcal{K})$ are both nonempty then

$$\big\{P_1 \cap \Lambda_n(\mathcal{K}), P_2 \cap \Lambda_n(\mathcal{K})\big\}$$

is a partition of $\Lambda_n(\mathcal{K})$ such that

$$\mathcal{U}(P_1 \cap \Lambda_n(\mathcal{K})) \cap \mathcal{U}(P_2 \cap \Lambda_n(\mathcal{K})) = \emptyset,$$

which we have just shown is not possible. It follows that for $i \in [1, 2]$, P_i is the union of the $\Lambda_n(\mathcal{K})$'s with which it has a nonempty intersection. This verifies the lemma. $\square$

It now follows from (3.12), (3.16), (3.17), and (3.19) that the

$$\text{density of } \Pi_+(\mathbf{a}, \mathbf{b}) = 2^{m-|\Sigma|}, \text{ whenever } \mathcal{M}_1 = \emptyset. \tag{3.21}$$

We turn next to the case when $\mathcal{M}_1 \neq \emptyset$, i.e., $\mathbf{b}$ has exactly one coordinate with square-free part 1, say σ_{i_0}, and $i_0 \in \Sigma$. It follows from the fact that Σ is the pairwise disjoint union of the supports $D_i, i \in [1, m]$ of the blocks of the quotient diagram that there exits a unique integer $n_0 \in [1, m]$ such that $i_0 \in D_{n_0}$, and so

$$\mathcal{M}_1 = \{I \in \Lambda_{n_0}(\mathcal{K}) : i_0 \in I\}, \tag{3.22}$$

$$\mathcal{M}_0 = \Big(\bigcup_{n \neq n_0} \Lambda_n(\mathcal{K})\Big) \cup \Big(\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1\Big). \tag{3.23}$$

In order to determine the value of ε in this case, we must determine when

$$\mathcal{U}(\mathcal{M}_0) \cap \mathcal{U}(\mathcal{M}_1) = \emptyset.$$

This is done in the next lemma.

Lemma 3.4. *The following statements are equivalent:*

(i) $\mathcal{U}(\mathcal{M}_0) \cap \mathcal{U}(\mathcal{M}_1) = \emptyset$;

(ii) $\mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K})$ *and* $\mathcal{M}_0 = \bigcup_{n \neq n_0} \Lambda_n(\mathcal{K})$;

(iii) $\mathcal{D}_{n_0}$ *has* 2 *or* 3 *rows, and in the latter instance, the row that is labeled by* i_0 *is the second row and the first and third rows do not overlap.*

Proof. Assume that *(i)* is true; then by (3.22) and (3.23),

$$\mathcal{U}(\mathcal{M}_1) \cap \mathcal{U}(\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1) = \emptyset.$$

If $\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 \neq \emptyset$, then $\{\mathcal{M}_1, \Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1\}$ is a partition of $\Lambda_{n_0}(\mathcal{K})$ whose existence contradicts Lemma 3.3. Hence

$$\mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K}).$$

When this is combined with (3.23), statement *(ii)* is obtained. That *(ii)* implies *(i)* is clear from (3.14). Because of (3.22), $\mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K})$ if and only if for every column C of $\mathcal{D}_{n_0}$ such

that $|C| \geq 2$, i_0 is a member of every element of $\mathcal{E}(\theta(C))$. That happens precisely when $\mathcal{D}_{n_0}$ has the structure as described in (iii). $\square$

We conclude from Lemma 3.4 that

$$\varepsilon = \begin{cases} 1, & \text{if } \mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K}), \\ 0, & \text{otherwise}. \end{cases} \tag{3.24}$$

We now observe that $\pi(\sigma_{i_0}) = \emptyset$ and so it follows that the set $\{v(\pi(\sigma_i)) : i \in \Sigma\} \setminus \{0\}$ is linearly independent over F and has cardinality $|\Sigma| - 1$. Hence

$$d = |\Sigma| - 1. \tag{3.25}$$

The elements of $\mathcal{P}_\emptyset(\mathcal{M}_0, 2)$ must now be counted.

Suppose first that $\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 \neq \emptyset$. Let $\{P_1, P_2\} \in \mathcal{P}_\emptyset(\mathcal{M}_0, 2)\}$. Then

$$\{\mathcal{M}_1 \cup P_1, P_2\} \in \mathcal{P}_\emptyset(\Lambda(\mathcal{K}), 2),$$

and so by Lemma 3.3, there exists a partition $\{Q_1, Q_2\}$ of $[1, m]$ such that

$$\mathcal{M}_1 \cup P_1 = \bigcup_{i \in Q_1} \Lambda_i(\mathcal{K}), \text{ and}$$

$$P_2 = \bigcup_{i \in Q_2} \Lambda_i(\mathcal{K}).$$

It follows that $n_0 \in Q_1$ and

$$P_1 = \Big(\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1\Big) \cup \Big(\bigcup_{i \in Q_1 \setminus \{n_0\}} \Lambda_i(\mathcal{K})\Big).$$

Because $\Lambda_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 \neq \emptyset$, it follows that there is a bijection of $\mathcal{P}_\emptyset(\mathcal{M}_0, 2)$ onto $\mathcal{P}([1, m], 2)$. Hence

$$\big|\mathcal{P}_\emptyset(\mathcal{M}_0, 2)\big| = 2^{m-1} - 1.$$

We conclude from this equation, (3.12), (3.24), and (3.25) that the

$$\text{density of } \Pi_+(\mathbf{a}, \mathbf{b}) = 2^{1-|\Sigma|}(2^m - 1), \text{ whenever } \emptyset \neq \mathcal{M}_1 \neq \Lambda_{n_0}(\mathcal{K}). \tag{3.26}$$

Suppose finally that $\mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K})$. Applying Lemma 3.3 to a partition $\{P_1, P_2\} \in \mathcal{P}_\emptyset(\mathcal{M}_0, 2)$ as before, we find a partition $\{Q_1, Q_2\}$ such that

$$n_0 \in Q_1 \text{ and } P_1 = \bigcup_{i \in Q_1 \setminus \{n_0\}} \Lambda_i(\mathcal{K}).$$

Because P_1 is nonempty, it follows that $Q_1 \setminus \{n_0\}$ is also nonempty. We conclude that there is a bijection of $\mathcal{P}_\emptyset(\mathcal{M}_0, 2)$ onto $\mathcal{P}([1, m-1], 2)$, and so

$$\big|\mathcal{P}_\emptyset(\mathcal{M}_0, 2)\big| = 2^{m-2} - 1.$$

It follows from this equation, (3.12), (3.24), and (3.25) that the

$$\text{density of } \Pi_+(\mathbf{a}, \mathbf{b}) = 2^{m-|\Sigma|}, \text{ whenever } \emptyset \neq \mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K}). \tag{3.27}$$

From (3.21), (3.26), and (3.27) comes the following theorem, one of the principal results of this paper:

Theorem 3.5. *If* $(\boldsymbol{a}, \boldsymbol{b})$ *is an admissible* $2k$*-tuple which satisfies condition* (3.9) *and for which the square-free parts of the coordinates of* $\boldsymbol{b}$ *are distinct, and if*

$$\mathcal{M}_1 = \{I \in \Lambda(\mathcal{K}) : 1 \in S(I)\},$$

$$\sigma = |\Sigma|,$$

$$m = \textit{the number of blocks in the quotient diagram of } (\boldsymbol{a},\ \boldsymbol{b}),$$

and n_0 *is the location of the block of the quotient diagram in whose support is located the index* i_0 *that determines* $\mathcal{M}_1$ *as per* (3.22) *, then the density of* $\Pi_+(\boldsymbol{a}, \boldsymbol{b})$ *is*

$$2^{m-\sigma},\ \textit{if } \mathcal{M}_1 = \emptyset \textit{ or } \emptyset \neq \mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K}),$$

or

$$2^{1-\sigma}(2^m - 1),\ \textit{if } \emptyset \neq \mathcal{M}_1 \neq \Lambda_{n_0}(\mathcal{K}).$$

Remark. Whenever $(\mathbf{a}, \mathbf{b})$ is an admissible $2k$-tuple which satisfies the hypotheses of Theorem 3.5 and $(D_1, \ldots, D_m)$ are the supports of the blocks in the quotient diagram then

$$\left|\Sigma\right| = \sum_{i=1}^{m} |D_i| \geq 2m.$$

Hence the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ is at most 2^{-m} whenever $\mathcal{M}_1 = \emptyset$ or $\emptyset \neq \mathcal{M}_1 = \Lambda_{n_0}(\mathcal{K})$, and is at most $(2^m - 1)/2^{2m-1}$, otherwise. This gives an interesting number-theoretic interpretation to the number of blocks in the quotient diagram. We also note that if $\mathbf{Q}$ denotes the rational numbers and $\mathcal{A}$ denotes the set of all admissible $2k$-tuples which satisfy the hypotheses of Theorem 3.5, $k \in [2, +\infty)$, and if we define the function $\Delta : \mathcal{A} :\to \mathbf{Q}$ by

$$\Delta(\mathbf{a}, \mathbf{b}) = \text{ the density of } \Pi_+(\mathbf{a}, \mathbf{b}),$$

then Lemma 2.5 can be used to show that the extremal values 2^{-m} and $(2^m - 1)/2^{2m-1}$ are each contained in the range of Δ; in fact, if a rational number q can be in the range of Δ, it will be, and for all such q, $\Delta^{-1}(q)$ is an infinite set.

We will now indicate how the previous argument can be modified so as to obtain the density if $\Pi_+(\mathbf{a}, \mathbf{b})$ when the square-free parts of the coordinates of $\mathbf{b}$ are not necessarily distinct. Let

$$\Sigma_i = D_i \cap \Sigma,$$
$$\delta = \{i : \Sigma_i \neq \emptyset\}.$$

If δ is empty then every element of $\{S(I) : I \in \Lambda(\mathcal{K})\}$ is a singleton, hence $\Pi_+(\mathbf{a}, \mathbf{b})$ is the set of all allowable primes, with consequent density 1. We hence assume that δ is not empty and choose an $n \in \delta$ that will remain fixed in the sequel.

We adopt the following convention, to be used in the rest of what follows; we identify an element I of $\Lambda'(\mathcal{K})$ with its subset $Z(I)$ and then, in an abuse of notation that we hope will not prove confusing, continue to denote that element by I. As $|S(I)| \geq 2$ for each $I \in \Lambda'(\mathcal{K})$, we may assume that the elements of

$$\Lambda'_n(\mathcal{K}) = \Lambda'(\mathcal{K}) \cap \Lambda_n(\mathcal{K})$$

consist precisely of all sets of cardinality at least 2 that are of the form $\Sigma_n \cap E$, where E varies throughout $\Lambda_n(\mathcal{K})$.

Our next task is the definition of a certain equivalence relation $\sim$ on Σ_n. If $(i,j) \in \Sigma_n \times \Sigma_n$ then we declare that $i \sim j$ if either $i = j$ or there is a subset $\{s_1 < \cdots < s_r\}$ of consecutive elements of Σ_n such that

$$\{i,j\} = \{s_1, s_r\}$$

and for each $i \in [1, r-1]$ there exists $C \in \mathcal{C}_n$ such that $\{s_i, s_{i+1}\} \subseteq \theta(C)$. When the square-free parts of the coordinates of $\mathbf{b}$ are distinct, then, as we showed in the proof of Lemma 3.3, this equivalence relation has only one equivalence class, namely $\Lambda_n(\mathcal{K})$. When the square-free parts of the coordinates of $\mathbf{b}$ are not distinct, the equivalence classes of $\sim$ divide the n-th block of the quotient diagram of $(\mathbf{a}, \mathbf{b})$ into "cells" in which can be applied the argument that was used above to count the elements of $\mathcal{P}_\emptyset(\mathcal{M}_0, 2)$. Our strategy now is to use this fact to count in a similar manner the elements of $\mathcal{P}_\emptyset(\mathcal{M}_0, 2)$ when square-free parts are no longer distinct. The next lemma records some technical facts about the equivalence classes of $\sim$ that will be of use in that endevour.

Lemma 3.6. (i) *Every equivalence class of* $\sim$ *has cardinality at least* 2.

(ii) *Every element of* $\Lambda'_n(\mathcal{K})$ *is contained in an equivalence class of* $\sim$ *and every equivalence class of* $\sim$ *contains an element of* $\Lambda'_n(\mathcal{K})$.

Proof. (i) Suppose that $\{i\}$ is an equivalence class of $\sim$. Consider the row in the block $\mathcal{D}_n$ of the quotient diagram of $(\mathbf{a}, \mathbf{b})$ that is labeled by i. Either there is a column C of cardinality at least 2 hanging from that row or that row is the *last* row of $\mathcal{D}_n$. In the former case there is an element i' of D_n such that i' is the smallest element of D_n exceeding i. Then $\{i, i'\} \subseteq \theta(C)$. If $\sigma_i \neq \sigma_{i'}$, then $\{i, i'\} \subseteq \Sigma_n$, whence $i \sim i'$, which is impossible as $\{i\}$ is an equivalence class of $\sim$. It follows that $\sigma_i = \sigma_{i'}$ and so in the construction of $\Lambda'(\mathcal{K})$, the set $\{i, i'\}$ was removed. Hence $i \notin \Sigma_n$, contrary to its choice. We conclude that no such element i can exist. In the latter case, there is a largest element i' of D_n that is less than i and there is a column C hanging from a row above the row labeled by i such that $\{i', i\} \subseteq \theta(C)$; otherwise, the last coordinate in the gap sequence of $\mathcal{D}_n$ will exceed $s-1$, which is impossible. We now argue as before to obtain yet again a contradiction to i's membership in Σ_n.

(ii) If $I \in \Lambda'_n(\mathcal{K})$ then there exists a column C of $\mathcal{D}_n$ and $E \in \mathcal{E}(\theta(C))$ such that $I = \Sigma_n \cap E$. Hence

$$[\min I, \max I] \cap D_n \subseteq [\min E, \max E] \cap D_n \subseteq \theta(C).$$

As $|I| \geq 2$, it follows that $\min I < \max I$, hence $\Sigma_n \cap [\min I, \max I]$ is contained in an equivalence class of $\sim$, and so therefore is I.

Every equivalence class of $\sim$, having cardinality at least 2, must contain at least two elements i and i' that are both contained in $\theta(C)$ for some column C of $\mathcal{D}_n$. Hence $\{i, i'\} \in \Lambda'_n(\mathcal{K})$. $\square$

Lemma 3.7. *If* $\Sigma_n/\sim$ *denotes the set of all equivalence classes of* $\sim$ *and* $\mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2)$ *denotes the set*

$$\big\{\{P_1, P_2\} \in \mathcal{P}(\Lambda'_n(\mathcal{K}), 2) : \ \mathcal{U}(P_1) \cap \mathcal{U}(P_2) = \emptyset\big\}$$

then there is a bijection of $\mathcal{P}(\Sigma_n/\sim, 2)$ *onto* $\mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2)$.

Proof. Let $\{P_1, P_2\} \in \mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2)$. Suppose that $\varpi \in \Sigma_n/\sim$ and $i \in \varpi \cap \mathcal{U}(P_1)$. We wish to show that $\varpi \subseteq \mathcal{U}(P_1)$, hence let $l \in \varpi$ with $l \neq i$. Then by definition of $\sim$, l can be connected to i by a chain $s_1 < \cdots < s_r$ of consecutive elements of Σ_n such that for

each $i \in [1, r-1]$, there exists $C \in \mathcal{C}_n$ such that $\{s_i, s_{i+1}\} \subseteq \theta(C)$. It follows that for each $i \in [1, r-1], \{s_i, s_{i+1}\} \in \Lambda'_n(\mathcal{K})$, and so an inductive argument similar to the one used in the proof of Lemma 3.3 using the facts that $i \in \mathcal{U}(P_1)$ and $\mathcal{U}(P_1) \cap \mathcal{U}(P_2) = \emptyset$ shows that each element of the chain, and hence l, is in $\mathcal{U}(P_1)$. When the same reasoning is applied to P_2, and when we also observe, by virtue of Lemma 3.6(*ii*), that every element of P_i is contained in an equivalence class of $\Sigma_n/\sim$, $i \in [1, 2]$, we conclude that $\mathcal{U}(P_i)$ is the union of the elements of $\Sigma_n/\sim$ with which it has a nonempty intersection, $i \in [1, 2]$. Set

$$E_i = \text{the set of elements of } \Sigma_n/\sim \text{ whose union is } \mathcal{U}(P_i), i \in [1, 2].$$

We claim that $\mathcal{U}(P_1) \cup \mathcal{U}(P_2) = \Sigma_n$; if this is true then $\{E_1, E_2\}$ is a partition of $\Sigma_n/\sim$. Let $i \in \Sigma_n$ and then choose $i' \neq i$ in the equivalence class of $\sim$ containing i such that there is a column C of $\mathcal{C}_n$ such that $\{i, i'\} \subseteq \theta(C)$). It follows that $\{i, i'\} \in \Lambda'_n(\mathcal{K})$, and so $\{i, i'\}$ must be in either P_1 or P_2, hence i is in either $\mathcal{U}(P_1)$ or $\mathcal{U}(P_2)$. This verifies our claim.

Conversely, suppose that $\{E_1, E_2\}$ partitions $\Sigma_n/\sim$. Set

$$P_i = \bigcup_{\varpi \in E_i} \{I \in \Lambda'_n(\mathcal{K}) : I \subseteq \varpi\}, \ i \in [1, 2].$$

It follows from Lemma 3.6(*ii*) and the fact that $\{E_1, E_2\}$ partitions $\Sigma_n/\sim$ that $\{P_1, P_2\} \in \mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2)$. Suppose now that $\varpi \in \Sigma_n/\sim, \varpi \cap \mathcal{U}(P_1) \neq \emptyset$, and $\varpi \notin E_1$. Then ϖ is disjoint from all the elements in E_1, and so is also disjoint from all the elements in P_1, contrary to the assumption that $\varpi \cap \mathcal{U}(P_1) \neq \emptyset$. Consequently $\mathcal{U}(P_1)$ is the union of the elements of E_1, and similar reasoning proves that $\mathcal{U}(P_2)$ is the union of the elements of E_2.

It is now straightforward to prove that the map $\mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2) \to \mathcal{P}(\Sigma_n/\sim, 2)$ defined in the first paragraph of this proof and the map $\mathcal{P}(\Sigma_n/\sim, 2) \to \mathcal{P}_\emptyset(\Lambda'_n(\mathcal{K}), 2)$ defined in the second paragraph are inverses of each other. $\square$

If $\varpi \in \Sigma_n/\sim$, then we will denote by $\Phi_n(\varpi)$ the subset of $\Lambda'_n(\mathcal{K})$ defined by

$$\{I \in \Lambda'_n(\mathcal{K}) : I \subseteq \varpi\},$$

and if

$$\varpi \in \bigcup_{i \in \delta} \left(\Sigma_i/\sim\right)$$

then there is a unique integer $i \in \delta$ such that $\varpi \in \Sigma_i/\sim$ and so we set $\Phi(\varpi)$ equal to $\Phi_i(\varpi)$. With this definition in place, Lemma 3.7 can be used in a straightforward modification of the proof of Lemma 3.3 to establish the following lemma, which is the replacement for Lemma 3.3 when there are repetitions of square-free parts.

Lemma 3.8. *The set $\mathcal{P}_\emptyset(\Lambda'(\mathcal{K}), 2)$ consists precisely of all sets of the form*

$$\Big\{ \bigcup_{\varpi \in E_1} \Phi(\varpi), \bigcup_{\varpi \in E_2} \Phi(\varpi) \Big\},$$

where $\{E_1, E_2\}$ varies throughout the set $\mathcal{P}\big(\bigcup_{i \in \delta} \left(\Sigma_i/\sim\right), 2\big)$.

When repetitions of square-free parts occur, we also need a replacement for Lemma 3.4 in our previous reasoning. We assume that $\mathcal{M}_1 \neq \emptyset$, and so for some $i_0 \in \Sigma, \sigma_{i_0} = 1$, and $\mathcal{D}_{n_0}$ is the block in the quotient diagram of $(\mathbf{a}, \mathbf{b})$ such that $i_0 \in D_{n_0}$. Then

$$\mathcal{M}_1 = \{I \in \Lambda'_{n_0}(\mathcal{K}) : i_0 \in I\}, \tag{3.28}$$

$$\mathcal{M}_0 = \Big(\bigcup_{n\neq n_0} \Lambda'_n(\mathcal{K})\Big) \cup \Big(\Lambda'_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1\Big), \tag{3.29}$$

as before. Let

$$\varpi_0 = \text{ the element of } \Sigma_{n_0}/\sim \text{ which contains } i_0.$$

The next two lemmas provide the replacement of Lemma 3.4 that we desire.

Lemma 3.9. $\mathcal{M}_1 \subseteq \Phi(\varpi_0)$.

Proof. Let C be a column of $\mathcal{D}_{n_0}$ of cardinality at least 2, let G be an element of $\mathcal{E}(\theta(C))$ such that $\big|\Sigma_{n_0} \cap G\big| \geq 2$, and suppose that $i_0 \in G$. We must prove that $\Sigma_{n_0} \cap G \subseteq \varpi_0$. In order to do that, observe that $\theta(C)$ consists of an interval of consecutive elements of D_{n_0}; consequently $\Sigma_{n_0} \cap \theta(C)$ consists of consecutive elements of Σ_{n_0}, and so $\Sigma_{n_0} \cap \theta(C)$ is contained in an equivalence class of $\sim$. Since $i_0 \in \theta(C)$, that equivalence class must be ϖ_0. Hence $\Sigma_{n_0} \cap G \subseteq \varpi_0$. $\square$

Lemma 3.10. *The sets $\mathcal{U}(\mathcal{M}_0)$ and $\mathcal{U}(\mathcal{M}_1)$ are disjoint if and only if either*

$$\mathcal{M}_1 = \Lambda'_{n_0}(\mathcal{K}),\ \mathcal{M}_0 = \bigcup_{n\neq n_0} \Lambda'_n(\mathcal{K}) \tag{3.30}$$

or $\Lambda'_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 \neq \emptyset$ and

$$\mathcal{M}_1 = \Phi(\varpi_0), \quad \mathcal{M}_0 = \Big(\Lambda'_{n_0}(\mathcal{K}) \setminus \Phi(\varpi_0)\Big) \cup \Big(\bigcup_{n\neq n_0} \Lambda'_n(\mathcal{K})\Big). \tag{3.31}$$

Proof. It follows from (3.14) as before that whenever $\mathcal{M}_0$ and $\mathcal{M}_1$ are determined according to (3.30), $\mathcal{U}(\mathcal{M}_0)$ and $\mathcal{U}(\mathcal{M}_1)$ are disjoint. In order to prove that this is also true when $\mathcal{M}_0$ and $\mathcal{M}_1$ are determined according to (3.31), it suffices to prove that

$$\mathcal{U}(\Phi(\varpi_0)) \cap \mathcal{U}(\Lambda'_{n_0}(\mathcal{K}) \setminus \Phi(\varpi_0)) = \emptyset,$$

and in order for this to hold, we need only prove that if $I \in \Lambda'_{n_0}(\mathcal{K}) \setminus \Phi(\varpi_0)$ then I is disjoint from every element in $\Phi(\varpi_0)$. But by virtue of Lemma 3.6(*ii*), I is contained in an equivalence class ϖ of $\Sigma_{n_0}/\sim$; as $I \notin \Phi(\varpi_0)$, it follows that $\varpi \neq \varpi_0$, and so every element of $\Phi(\varpi_0)$ must be disjoint from I.

Assume that $\mathcal{U}(\mathcal{M}_0)$ and $\mathcal{U}(\mathcal{M}_1)$ are disjoint and suppose that $\Lambda'_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 \neq \emptyset$. Then

$$\{\mathcal{M}_1,\ \Lambda'_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1\} \in \mathcal{P}_\emptyset(\Lambda'_{n_0}(\mathcal{K}),\ 2),$$

hence by virtue of Lemma 3.7, there exists a partition $\{E_1, E_2\}$ of $\Sigma_{n_0}/\sim$ such that

$$\mathcal{M}_1 = \bigcup_{\varpi\in E_1} \Phi(\varpi),\ \Lambda'_{n_0}(\mathcal{K}) \setminus \mathcal{M}_1 = \bigcup_{\varpi\in E_2} \Phi(\varpi). \tag{3.32}$$

It is now a consequence of (3.28) and the first equation in (3.32) that $E_1 = \{\varpi_0\}$. Hence $\mathcal{M}_1 = \Phi(\varpi_0)$, and so (3.31) follows from (3.29). $\square$

We now let

$$\mu = \Big|\bigcup_{i\in\delta} (\Sigma_i/\sim)\Big| = \sum_{i\in\delta} \big|\Sigma_i/\sim\big|,$$

and then use Lemmas 3.8-3.10 in a straightforward modification of our previous argument to deduce that

$$\varepsilon = \begin{cases} 1, \text{ if either } \mathcal{M}_1 = \emptyset \text{ or } \emptyset \neq \mathcal{M}_1 = \Phi(\varpi_0), \\ 0, \text{ otherwise,} \end{cases} \tag{3.33}$$

$$|\mathcal{P}_\emptyset(, \mathcal{M}_0, 2)| = \begin{cases} 2^{\mu-1} - 1, \text{ if } \mathcal{M}_1 = \emptyset \text{ or } \emptyset \neq \mathcal{M}_1 \neq \Phi(\varpi_0), \\ 2^{\mu-2} - 1, \text{ if } \emptyset \neq \mathcal{M}_1 = \Phi(\varpi_0). \end{cases} \tag{3.34}$$

We obtain from (3.17) and (3.25) that

$$d = \begin{cases} |\Sigma|, \text{ if } \mathcal{M}_1 = \emptyset\ , \\ |\Sigma| - 1, \text{ if } \mathcal{M}_1 \neq \emptyset. \end{cases} \tag{3.35}$$

Finally, a careful inspection of our previous argument reveals that instead of (3.9), we need only assume that the set $\{v(\pi(\sigma_i)) : i \in \Sigma\} \setminus \{0\}$ is linearly independent over F, and this is true if and only if

(3.36)
the product of all the elements in any nonempty subset of $\{\sigma_i : i \in \Sigma\} \setminus \{1\}$ is never a square.

Another run through our previous reasoning using (3.33)-(3.36) at the appropriate juncture hence verifies the following theorem, our second principal result.

Theorem 3.11. *Let* $(\mathbf{a}, \mathbf{b})$ *be an admissible* $2k$*-tuple and assume that the square-free parts of the coordinates of* $\mathbf{b}$ *satisfy* (3.36). *Let*

$$\mathcal{M}_1 = \{I \in \Lambda'(\mathcal{K}) : 1 \in S(I)\},$$

$$\sigma = |\Sigma|, \text{ and}$$

$$\mu = \sum_{i \in \delta} \left|\Sigma_i / \sim\right|.$$

If n_0 *is the location of the block of the quotient diagram of* $(\mathbf{a}, \mathbf{b})$ *in whose support is located the index* i_0 *that determines* $\mathcal{M}_1$ *as per* (3.28), *and* ϖ_0 *is the equivalence class of* $\Sigma_{n_0}/\sim$ *which contains* i_0, *then the density of* $\Pi_+(\mathbf{a}, \mathbf{b})$ *is*

$$2^{\mu-\sigma}, \text{ if } \mathcal{M}_1 = \emptyset \text{ or } \emptyset \neq \mathcal{M}_1 = \Phi(\varpi_0),$$

or

$$2^{1-\sigma}(2^\mu - 1), \text{ if } \emptyset \neq \mathcal{M}_1 \neq \Phi(\varpi_0).$$

Theorem 3.11 shows that when the equivalence relations $\sim$ on the Σ_i's disconnect blocks in the quotient diagram of $(\mathbf{a}, \mathbf{b})$, the cells of cardinality at least 2 that are formed from the disconnections increase the density of $\Pi_+(\mathbf{a}, \mathbf{b})$, with each cell contributing essentially a factor of 2 to the magnitude of the density. Upper bounds on the size of the density similar to those obtained before are also valid here. If $\mu = 0$ then $\Sigma = \emptyset$, i.e., for all $I \in \Lambda(\mathcal{K})$, $|S(I)| = 1$. It follows that $\Pi_+(\mathbf{a}, \mathbf{b})$ is the set of all allowable primes, with density 1. We note incidentally that $|S(I)| = 1$ for all $I \in \Lambda(\mathcal{K})$ if and only if $\prod_{i \in I} b_i$ is a square for all $I \in \Lambda(\mathcal{K})$, and this is valid, as we reported in the introduction, if and only if $q_\varepsilon(p) \sim (b \cdot 2^\kappa)^{-1} p$

as $p \to +\infty$. If $\mu \geq 1$ then by Lemma 3.6(i), $|\Sigma| \geq 2\mu$ and so the density is at most $2^{-\mu}$ if $1 \notin \{\sigma_i : i \in \Sigma\}$, and at most $(2^\mu - 1)/2^{2\mu-1}$, otherwise.

We conclude our discussion in this paper by indicating how Theorem 3.11 needs to be modified so as to obtain the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ for an arbitrary admissible $2k$-tuple $(\mathbf{a}, \mathbf{b})$, modulo the solution of a problem in enumerative combinatorics that we do not solve. In order to explain these modifications efficiently, the following notation will be useful. Let

$$E = \bigcup_{i \in \delta} \big(\Sigma_i / \sim\big),$$

and for an element ϖ of E, we set

$$\mathcal{S}(\varpi) = \bigcup_{I \in \Phi(\varpi)} \mathcal{S}(I).$$

We now define two combinatorial parameters α and β. First, consider the subset of $\mathcal{P}(E, 2)$ consisting of all elements $\{E_1, E_2\}$ which satisfy the following condition: there exists a subset U of

$$\big\{\pi(\sigma_i) : i \in \Sigma\big\} \cup \{\emptyset\}$$

of odd cardinality such that the cardinality of

$$U \cap \Big(\Big(\bigcup_{\varpi \in E_1} \mathcal{S}(\varpi)\Big) \cup \{\emptyset\}\Big)$$

is even and the repeated symmetric difference of the elements of U is empty. Let α denote the cardinality of this subset. Secondly, consider the subset of $\mathcal{P}(E \setminus \{\varpi_0\}, 2)$ consisting of all elements $\{E_1, E_2\}$ which satisfy the following condition: there exists a subset U of

$$\big\{\pi(\sigma_i) : i \in \Sigma\big\} \cup \{\emptyset\}$$

of odd cardinality such that the cardinality of

$$U \cap \Big(\mathcal{S}(\varpi_0) \cup \Big(\bigcup_{\varpi \in E_1} \mathcal{S}(\varpi)\Big) \cup \{\emptyset\}\Big)$$

is even and the repeated symmetric difference of the elements of U is empty. Let β denote the cardinality of this subset.

Now, for a general admissible $2k$-tuple $(\mathbf{a}, \mathbf{b})$, let

$$\varepsilon = \begin{cases} 1, \text{ if } \mathcal{N}(\mathcal{M}_0, \mathcal{M}_1) \neq \emptyset, \\ \quad 0, \text{ otherwise,} \end{cases}$$

$$n = \Big|\bigcup_{i \in \Sigma} \pi(\sigma_i)\Big|,$$

$$d = \text{ the dimension of the linear span of } \{v(\pi(\sigma_i)) : i \in \Sigma\} \setminus \{0\} \text{ in } F^n,$$

and let $\mathcal{M}_1, \mu$, and ϖ_0 be as defined in the statement of Theorem 3.11. Then the density of $\Pi_+(\mathbf{a}, \mathbf{b})$ is

$$2^{-d}(2^\mu - 2\alpha + \varepsilon - 1), \text{ if } \mathcal{M}_1 = \emptyset \text{ or } \emptyset \neq \mathcal{M}_1 \neq \Phi(\varpi_0),$$

$$2^{-d}(2^{\mu-1} - 2\beta + \varepsilon - 1), \text{ if } \emptyset \neq \mathcal{M}_1 = \Phi(\varpi_0).$$

We note that $\varepsilon = 1$ here if and only if $\mathcal{M}_0$ and $\mathcal{M}_1$ have the structure as specified in Lemma 3.10 and for all subsets U of

$$\{\pi(\sigma_i) : i \in \Sigma\} \cup \{\emptyset\}$$

of odd cardinality such that the cardinality of

$$U \cap \Big(\bigcup_{I \in \mathcal{M}_1} \mathcal{S}(I) \cup \{\emptyset\} \Big)$$

is even, the repeated symmetric difference of the elements of U is not empty.

When $(\mathbf{a}, \mathbf{b})$ satisfies condition (3.36), $\alpha = \beta = 0$, hence Theorem 3.11 gives "local" maximum values of the density of $\Pi_+(\mathbf{a}, \mathbf{b})$, and also the location of these local maxima, as $(\mathbf{a}, \mathbf{b})$ varies throughout all admissible $2k$-tuples, $k \in [2, +\infty)$. The calculation of the parameters α and β is an interesting problem in enumerative combinatorics that we invite the curious reader to contemplate.

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